EXPONENTIAL AND GAUSSIAN CONCENTRATION OF 1-LIPSCHITZ MAPS

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ABSTRACT. In this paper, we prove an exponential and Ganssian concentration inequality for 1-Lipschitz maps from mm-spaces to Hadamard manifolds. In particular, we give a complete answer to a question by M. Gromov.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this paper, we study the theory of the Lévy-Milman concentration of 1-Lipschitz maps from an mm-space (metric measure space) into an Hadamard manifold. An *mm-space* $X = (X, d_X, \mu_X)$ is a complete separable metric space (X, d_X) with a Borel probability measure μ_X. Let N be an m-dimensional Hadamard manifold, i.e., a complete simply-connected Riemannian manifold with non-positive sectional curvature. Given a Borel measurable map $f : X \to N$ such that the push-forward measure $f_*(\mu_X)$ of μ_X by f has the finite moment of order 2, we define its *expectation* $\mathbb{E}(f)$ by the center of mass of the measure $f_*(\mu_X)$.

We shall consider a closed Riemannian manifold M as an mm-space with the volume measure μ_M normalized as $\mu_M(M) = 1$. We denote by $\lambda_1(M)$ the first non-zero eigenvalue of the Laplacian on M. In [5, Section $3\frac{1}{2}$.41], M. Gromov proved that

$$\mu_M(\{x \in M \mid d_N(f(x), \mathbb{E}(f)) \geq r\}) \leq m/(\lambda_1(M)r^2) \tag{1.1}$$

for any 1-Lipschitz map $f : M \to N$, where N is any m-dimensional Hadamard manifold. He also asked in [5, Section $3\frac{1}{2}$.41] that if the right-hand side of the above inequality (1.1) can be improved by the form $C_1 e^{-C_2\sqrt{m/\lambda_1(M)}r}$ or the form $C_1 e^{-C_2(m/\lambda_1(M))r^2}$. In this paper, we give an answer to this question affirmatively.

To state our main result, we need some definition. We define the *concentration function* $\alpha_X : (0, +\infty) \to \mathbb{R}$ of an mm-space X as the supremum of $\mu_X(X \setminus A_r)$, where A runs over all Borel subsets of X with $\mu_X(A) \geq 1/2$ and A_r is an open r-neighbourhood of A. We shall consider an mm-space X satisfying that

$$\alpha_X(r) \leq C_X e^{-c_X r^p} \tag{1.2}$$

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for any $r > 0$ and some constants $c_X, C_X, p > 0$. In the case of $p = 1$ (resp., $p = 2$), the space X is said to have the *exponential concentration* (resp., *Gaussian concentration*). For example, a closed Riemannian manifold M satisfies that $\alpha_M(r) \leq e^{-\sqrt{\lambda_1(M)}r/3}$ ([2, Theorem 4.1], [7, Theorem 3.1]). If the manifold M moreover satisfies that $Ric_M \geq \kappa > 0$, then we have $\alpha_M(r) \leq e^{-\kappa r^2/2}$ ([2, Section 1.2, Remark 2], [7, Theorem 2.4]). For an mm-space satisfying (1.2) and $m \in \mathbb{N}$, we put

$$A_{m,X} := 1 + \frac{\sqrt{\pi}}{4} \max\{1, 2C_X\} e^{2C_X + (m+1)/(4m-2)} \{2 + e^{1/(4m-2)}\}$$

and

$$\widetilde{A}_{m,X} := 1 + \frac{\sqrt{\pi} C_X e^{(m+1)/(4m-2)}}{2} \{2 + e^{1/(4m-2)}\}.$$

We also put

$$B_{m,X} := 1 + \frac{\sqrt{\pi} e^{(m+1)/(4m-2)}}{2} \max\{e^{(\pi C_X)^2/2}, 2C_X e^{(\pi C_X)^2}\}$$

and

$$\widetilde{B}_{m,X} := 1 + \sqrt{\pi} C_X e^{(m+1)/(4m-2)}.$$

Our main result is the following.

Theorem 1.1. *Let an mm-space X satisfy (1.2), N be an m-dimensional Hadamard manifold, and $f : X \to N$ a 1-Lipschitz map. Then, we have the following* (1) *and* (2)*.*

(1) *If $p = 1$, then, for any $r > 0$, we have*

$$\mu_X(\{x \in X \mid d_N(f(x), \mathbb{E}(f)) \geq r\}) \leq \min\{A_{m,X} e^{-(c_X/\sqrt{2m})r}, \widetilde{A}_{m,X} e^{-(c_X/(2\sqrt{2m}))r}\}. \tag{1.3}$$

(2) *If $p = 2$, then, for any $r > 0$, we have*

$$\mu_M(\{x \in X \mid d_N(f(x), \mathbb{E}(f)) \geq r\}) \leq \min\{B_{m,X} e^{-(c_X/(8m))r^2}, \widetilde{B}_{m,X} e^{-(c_X/(16m))r^2}\}. \tag{1.4}$$

As a corollary of Theorem 1.1, we obtain the following. For $m \in \mathbb{N}$, we put

$$A_m := 1 + \frac{\sqrt{\pi} e^{(9m-3)/(4m-2)}}{2} \{2 + e^{1/(4m-2)}\} \text{ and } \widetilde{A}_m := 1 + \frac{\sqrt{\pi} e^{(m+1)/(4m-2)}}{2} \{2 + e^{1/(4m-2)}\}.$$

We also put

$$B_m := 1 + \sqrt{\pi} e^{\pi^2 + (m+1)/(4m-2)} \text{ and } \widetilde{B}_m := 1 + \sqrt{\pi} e^{(m+1)/(4m-2)}.$$

Corollary 1.2. *Let M be a closed Riemannian manifold, N an m-dimensional Hadamard manifold, and $f : M \to N$ a 1-Lipschitz map. Then, we have the following* (1) *and* (2)*.*

(1) *For any $r > 0$, we have*

$$\begin{aligned} &\mu_M(\{x \in M \mid d_N(f(x), \mathbb{E}(f)) \geq r\}) \\ \leq\ &\min\{A_m e^{-3^{-1}\sqrt{\lambda_1(M)/(2m)}r}, \widetilde{A}_m e^{-6^{-1}\sqrt{\lambda_1(M)/(2m)}r}\}. \end{aligned} \tag{1.5}$$

(2) *If moreover $Ric_M \geq \kappa > 0$ holds, then for any $r > 0$ we also have*

$$\mu_M(\{x \in M \mid d_N(f(x), \mathbb{E}(f)) \geq r\}) \leq \min\{B_m e^{-(\kappa/(16m))r^2}, \widetilde{B}_m e^{-(\kappa/(32m))r^2}\}. \tag{1.6}$$

The inequality (1.5) is sharper than the inequality (1.1) if r is large enough. In the case where $M = \mathbb{S}^n$, the inequality (1.6) is sharp in a sense (see Remark 4.5). Theorem 1.1 and Corollary 1.2 answer the question by Gromov. To prove the theorem, we use a traditional method of the Gibbs-Laplace transform (see [7, Section 1.6]), i.e., we estimate $\int_X e^{\lambda\, d_N(f(x),\mathbb{E}(f))} d\mu_X(x)$ for $\lambda > 0$ from above, and then substitute a suitable value to λ. To do this, we estimate $\int_X d_N(f(x), \mathbb{E}(f))^q d\mu_X(x)$ for $q \geq 1$ by using the method of M. Ledoux and K. Oleszkiewicz in [8, Theorem 1].

For $m \leq n$, we consider the m-dimensional standard unit sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$ centered at zero as a subset of $\mathbb{S}^n$ in a natural way. As an application of Corollary 1.2, we estimate $\mu_{\mathbb{S}^n}(\mathbb{S}^n \setminus (\mathbb{S}^m)_r)$ from above (Corollary 4.1). In [1], S. Artstein studied an asymptotic behavior of the values $\mu_{\mathbb{S}^n}(\mathbb{S}^n \setminus (\mathbb{S}^m)_r)$. We will compare our estimate with those Artstein's results (see Remark 4.5). Denote by γ_m the standard Gaussian measure on $\mathbb{R}^m$ with the density $(2\pi)^{-m/2} e^{-|x|^2/2}$. In [8, Theorem 1], motivated by the work of Gromov ([4]), Ledoux and Oleszkiewicz obtained that if an mm-space having the Gaussian concentration (1.2), then for an m-dimensional Hadamard manifold N and a 1-Lipschitz map $f : X \to N$, we have

$$\mu_X(\{x \in X \mid d_N(f(x), \mathbb{E}(f)) \geq r\}) \leq C C_X \gamma_m(\{x \in \mathbb{R}^m \mid |x| \geq C\sqrt{c_X} r\}), \tag{1.7}$$

where $C > 0$ is a universal constant. Their estimate (1.7) is highly relevant with our two estimate in Theorem 1.1. We will compare these estimate (see Remark 4.6).

2. Preliminaries

2.1. **Concentration of 1-Lipschitz functions around the expetcations.** In this subsection we explain some basic facts on the theory of the Lévy-Milman concentration of 1-Lipschitz functions, which will be useful to prove the main theorem. The theory of the concentration of 1-Lipschitz functions was introduced by V. Milman in his investigations of asymptotic geometric analysis ([10, 11, 12]).

Let X be an mm-space and $f : X \to \mathbb{R}$ a Borel measurable function. A number $m_f \in \mathbb{R}$ is called a *median* of f if it satisfies that $\mu_X(\{x \in X \mid f(x) \geq m_f\}) \geq 1/2$ and $\mu_X(\{x \in X \mid f(x) \leq m_f\}) \geq 1/2$. We remark that m_f does exist, but it is not unique for f in general.

Lemma 2.1 ([7, Section 1.3]). *Let X be an mm-space. Then, for any 1-Lipschitz function $f : X \to \mathbb{R}$ and median m_f of f, we have*

$$\mu_X(\{x \in X \mid |f(x) - m_f| \geq r\}) \leq 2\alpha_X(r).$$

Conversely, if a function $\alpha : (0, +\infty) \to [0, +\infty)$ satisfies that

$$\mu_X(\{x \in X \mid |f(x) - m_f| \geq r\}) \leq \alpha(r)$$

for any 1*-Lipschitz function* $f : X \to \mathbb{R}$ *and median* m_f *of* f*, then we have*

$$\alpha_X(r) \leq \alpha(r).$$

Although the following lemma is stated in [7], we prove them for the completeness of this paper. Given $p > 0$, we put $K_p := \int_0^{+\infty} e^{-r^p} dr = \frac{1}{p}\Gamma\big(\frac{1}{p}\big)$.

Lemma 2.2 (cf. [7, Proposition 1.8])**.** *Assume that an mm-space* X *satisfies (1.2). Then, for any* $p \geq 1$ *and any* 1*-Lipschitz function* $f : X \to \mathbb{R}$ *with expectation zero, we have*

$$\mu_X(\{x \in X \mid |f(x)| \geq r\}) \leq \max\{e^{2(C_X K_p)^p}, 2C_X e^{(2C_X K_p)^p}\} e^{-2^{1-p} c_X r^p}.$$

Proof. By virtue of Lemma 2.1, we have

$$\mu_X(\{x \in X \mid |f(x) - m_f| \geq r\}) \leq 2C_X e^{-c_X r^p} \tag{2.1}$$

for any $r > 0$. By using this, we calculate

$$\begin{aligned} |m_f| &\leq \int_X |f(x) - m_f| d\mu_X(x) \\ &\leq \int_0^{+\infty} \mu_X(\{x \in X \mid |f(x) - m_f| \geq r\}) dr \\ &\leq 2C_X \int_0^{+\infty} e^{-c_X r^p} dr \\ &= \frac{2C_X K_p}{(c_X)^{1/p}} =: \overline{\alpha} \end{aligned} \tag{2.2}$$

If $r > \overline{\alpha}$, combining (2.1) with (2.2), we then get

$$\begin{aligned} \mu_X(\{x \in X \mid |f(x)| \geq r\}) \leq 2C_X e^{-c_X (r - \overline{\alpha})^p} &\leq 2C_X e^{-c_X 2^{1-p} r^p + c_X \overline{\alpha}^p} \\ &\leq 2C_X e^{(2C_X K_p)^p} e^{-c_X 2^{1-p} r^p}. \end{aligned}$$

If $r \leq \overline{\alpha}$, we then obtain

$$\begin{aligned} \mu_X(\{x \in X \mid |f(x)| \geq r\}) \leq 1 = e^{2(C_X K_p)^p} e^{-2(C_X K_p)^p} &= e^{2(C_X K_p)^p} e^{-2^{1-p} c_X \overline{\alpha}^p} \\ &\leq e^{2(C_X K_p)^p} e^{-2^{1-p} c_X r^p}. \end{aligned}$$

This completes the proof. □

2.2. **Expectation of a map to an Hadamard manifold.** In this subsection we define the expectation of a Borel measurable map from an mm-space to an Hadamard manifold. In order to define the expectation, we first explain some basic facts on the barycenter of a Borel probability measure on an Hadamard manifold.

Let N be an Hadamard manifold. We denote by $\mathcal{P}^2(N)$ the set of all Borel probability measure ν on N having the finite moment of order 2, i.e.,

$$\int_N d_N(x, y)^2 d\nu(y) < +\infty$$

for some (hence all) $x \in N$. A point $x_0 \in N$ is called the *barycenter* of a measure $\nu \in \mathcal{P}^2(N)$ if x_0 is the unique minimizing point of the function

$$N \ni x \mapsto \int_N d_N(x,y)^2 d\nu(y) \in \mathbb{R}.$$

We denote the point x_0 by $b(\nu)$. It is well-known that every $\nu \in \mathcal{P}^2(N)$ has the barycenter ([13, Proposition 4.3]).

A simple variational argument implies the following two lemmas.

Lemma 2.3 (cf. [13, Proposition 5.4])**.** *For each $\nu \in \mathcal{P}^2(\mathbb{R}^m)$, we have*

$$b(\nu) = \int_{\mathbb{R}^m} y d\nu(y).$$

Lemma 2.4 (cf. [13, Proposition 5.10])**.** *Let N be an Hadamard manifold and $\nu \in \mathcal{P}^2(N)$. Then $x = b(\nu)$ if and only if*

$$\int_N \exp_x^{-1}(y) d\nu(y) = 0.$$

In particular, identifying the tangent space of N at $b(\nu)$ with the Euclidean space of the same dimension, we have $b((\exp_{b(\nu)}^{-1})_*(\nu)) = 0$.

Let $f : X \to N$ be a Borel measurable map from an mm-space X to an Hadamard manifold N satisfying $f_*(\mu_X) \in \mathcal{P}^2(N)$. We define the *expectation* $\mathbb{E}(f) \in N$ of the map f by the point $b(f_*(\mu_X))$. By Lemma 2.3, in the case where N is a Euclidean space, this definition coincides with the classical one:

$$\mathbb{E}(f) = \int_X f(x) d\mu_X(x).$$

3. Proof of the main theorem

Let X be an mm-space satisfying (1.2) and $f : X \to \mathbb{R}^m$ a 1-Lipschitz map with expectation zero. To prove the main theorem, we shall estimate $V_q(f) := (\int_X |f(x)|^q d\mu_X(x))^{1/q}$ and $\widetilde{V}_q(f) := (\int_{X\times X} |f(x) - f(y)|^q d(\mu_X \times \mu_X)(x,y))^{1/q}$ for $q \geq 1$. We show Ledoux and Oleskiewicsz's argument in [8, Theorem 1] as follows.

Let $\varphi : X \to \mathbb{R}$ be an arbitrary 1-Lipschitz function with expectation zero and $q \geq 1$. For any $\alpha > -1$, we put

$$M_\alpha := \int_{\mathbb{R}} |s|^\alpha d\gamma_1(s) = 2^{\alpha/2}\pi^{-1/2}\Gamma\Big(\frac{\alpha+1}{2}\Big).$$

By virtue of Lemma 2.2, we obtain $\mu_X(\{x \in X \mid |\varphi(x)| \geq r\}) \leq C_1 e^{-C_2 r^p}$, where both C_1 and C_2 are defined by

$$C_1 := \max\{e^{2(C_X K_p)^p}, 2C_X e^{(2C_X K_p)^p}\} \text{ and } C_2 := 2^{1-p} c_X.$$

We calculate that

$$
\begin{aligned}
\int_X |\varphi(x)|^q d\mu_X(x) &= \int_0^{+\infty} \mu_X(\{x \in X \mid |\varphi(x)| \geq r\}) d(r^q) \\
&\leq C_1 \int_0^{+\infty} e^{-C_2 r^p} d(r^q) \\
&= \frac{\sqrt{2\pi} q C_1 M_{\frac{2q}{p}-1}}{p(2C_2)^{q/p}}.
\end{aligned} \tag{3.1}
$$

Given any 1-Lipschitz map $f : X \to \mathbb{R}^m$ with expectation zero and $z \in \mathbb{R}^m$, the map $z \cdot f : X \to \mathbb{R}$ is the $|z|$-Lipschitz function with expectation zero. By using the inequality (3.1), we hence have

$$
\begin{aligned}
V_q(f)^q &= \int_X \Big\{ \frac{1}{M_q} \int_{\mathbb{R}^m} |z \cdot f(x)|^q d\gamma_m(z) \Big\} d\mu_X(x) \\
&\leq \frac{\sqrt{2\pi} q C_1 M_{\frac{2q}{p}-1}}{p(2C_2)^{q/p} M_q} \int_{\mathbb{R}^m} |z|^q d\gamma_m(z)
\end{aligned}
$$

We therefore obtain

$$
V_q(f)^q \leq \frac{2^{-(q/p)+(q/2)} \sqrt{\pi} \max\{e^{2(C_X K_p)^p}, 2C_X e^{(2C_X K_p)^p}\}}{(c_X)^{q/p}} \cdot \frac{q\Gamma\left(\frac{q}{p}\right)}{p\Gamma\left(\frac{q+1}{2}\right)} \int_{\mathbb{R}^m} |z|^q d\gamma_m(z). \tag{3.2}
$$

To get another estimate, we repeat the above argument by using the following lemma.

Lemma 3.1 (cf. [7, Corollary 1.5])**.** *Let X be an mm-space and $\varphi : X \to \mathbb{R}$ a 1-Lipschitz function. Then, for any $r > 0$, we have*

$$
(\mu_X \times \mu_X)(\{(x,y) \in X \times X \mid |\varphi(x) - \varphi(y)| \geq r\}) \leq 2\alpha_X(r/2).
$$

Let X, $\varphi : X \to \mathbb{R}$, and $f : X \to \mathbb{R}^m$ be as above. By Lemma 3.1, we calculate that

$$
\begin{aligned}
\widetilde{V}_q(\varphi)^q &= \int_0^{+\infty} (\mu_X \times \mu_X)(\{(x,y) \in X \times X \mid |\varphi(x) - \varphi(y)| \geq r\}) d(r^q) \\
&\leq 2C_X \int_0^{+\infty} e^{-c_X 2^{-p} r^p} d(r^q) \\
&= \frac{\sqrt{\pi} q 2^{q+(3/2)-(q/p)} C_X M_{\frac{2q}{p}-1}}{p(c_X)^{q/p}}.
\end{aligned}
$$

We hence get

$$
\begin{aligned}
\widetilde{V}_q(f)^q &= \int_{X\times X} \Big\{ \frac{1}{M_q} \int_{\mathbb{R}^m} |z\cdot (f(x)-f(y))|^q d\gamma_m(z) \Big\} d(\mu_X \times \mu_X)(x,y) \\
&\le \frac{\sqrt{\pi} q 2^{q+(3/2)-(q/p)} C_X M_{\frac{2q}{p}-1}}{p(c_X)^{q/p} M_q} \int_{\mathbb{R}^m} |z|^q d\gamma_m(z) \\
&= \frac{\sqrt{\pi} 2^{(q/2)+1} C_X}{(c_X)^{q/p}} \cdot \frac{q\Gamma\left(\frac{q}{p}\right)}{p\Gamma\left(\frac{q+1}{2}\right)} \int_{\mathbb{R}^m} |z|^q d\gamma_m(z).
\end{aligned} \tag{3.3}
$$

Since $V_q(f) \le \widetilde{V}_q(f)$, we therefore obtain

$$
V_q(f)^q \le \frac{\sqrt{\pi} 2^{(q/2)+1} C_X}{(c_X)^{q/p}} \cdot \frac{q\Gamma\left(\frac{q}{p}\right)}{p\Gamma\left(\frac{q+1}{2}\right)} \int_{\mathbb{R}^m} |z|^q d\gamma_m(z). \tag{3.4}
$$

Remark 3.2. We shall compare the inequality (3.2) with the inequality (3.4). For fixed p, c_X, C_X, the inequality (3.4) is worse than the inequality (3.2) if q is large enough. If we fix q, c_X, then the inequality (3.4) is sharper than the inequality (3.2) if p or C_X is large enough.

We next explain the following observation by Gromov.

Proposition 3.3 (cf. [3, Section 13])**.** *Let $f : X \to N$ be a 1-Lipschitz map from an mm-space X to an m-dimensional Hadamard manifold such that $f_*(\mu_X) \in \mathcal{P}^2(N)$. We identify the tangent space at $\mathbb{E}(f)$ with the Euclidean space $\mathbb{R}^m$ and consider the map $f_0 := \exp_{\mathbb{E}(f)}^{-1} \circ f : X \to \mathbb{R}^m$. Then, the map f_0 is a 1-Lipschitz map with expectation zero satisfying that*

$$
\mu_X(\{x \in X \mid d_N(f(x), \mathbb{E}(f)) \ge r\}) = \mu_X(\{x \in X \mid |f_0(x)| \ge r\}) \tag{3.5}
$$

for any $r > 0$.

Proof. The 1-Lipschitz continuity of the map f_0 follows from Toponogov's comparison theorem. By Lemma 2.4, the expectation of the map f_0 is zero. Since the map $\exp_{\mathbb{E}(f)}^{-1}$ is isometric on rays issuing from $\mathbb{E}(f)$, we obtain (3.5). This completes the proof. □

Proof of Theorem 1.1. According to Proposition 3.3, we only prove the case of $N = \mathbb{R}^m$. For $p = 1, 2$, we put

$$
(C_1, C_2) := \Big(\frac{2\sqrt{\pi} C_X}{p}, \frac{\sqrt{2}}{(c_X)^{1/p}} \Big) \text{ or } \Big(\frac{\sqrt{\pi}}{p} \max\{ e^{2(C_X K_p)^p}, 2C_X e^{(2C_X K_p)^p} \}, \frac{2^{-(1/p)+(1/2)}}{(c_X)^{1/p}} \Big).
$$

Let $f : X \to \mathbb{R}^m$ be an arbitrary 1-Lipschitz map with expectation zero.

Assuming that $p=1$, we first prove the inequality (1.3). According to the inequalities (3.2) and (3.4), for $\lambda > 0$, we estimate

$$\begin{aligned}\int_X e^{\lambda|f(x)|}d\mu_X(x) &= 1+\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}V_k(f)^k \\ &\leq 1+C_1C_2\lambda\sum_{k=1}^{\infty}\frac{(C_2\lambda)^{k-1}}{\Gamma\big(\frac{k+1}{2}\big)}\int_{\mathbb{R}^m}|z|^k d\gamma_m(z) \\ &= 1+C_1C_2\lambda\int_{\mathbb{R}^m}|z|\sum_{k=0}^{\infty}\frac{(C_2\lambda|z|)^k}{\Gamma\big(\frac{k}{2}+1\big)}d\gamma_m(z).\end{aligned}$$

Since

$$\sum_{k=0}^{\infty}\frac{(C_2\lambda|z|)^k}{\Gamma\big(\frac{k}{2}+1\big)}=\sum_{k=0}^{\infty}\frac{(C_2\lambda|z|)^{2k}}{\Gamma(k+1)}+\sum_{k=0}^{\infty}\frac{(C_2\lambda|z|)^{2k+1}}{\Gamma\big(\frac{2k+1}{2}+1\big)}\leq e^{(C_2\lambda)^2|z|^2}+C_2\lambda|z|e^{(C_2\lambda)^2|z|^2},$$

we thus get

$$\begin{aligned}&\int_X e^{\lambda|f(x)|}d\mu_X(x) \\ &\leq 1+C_1C_2\lambda\int_{\mathbb{R}^m}|z|e^{(C_2\lambda)^2|z|^2}d\gamma_m(z)+C_1(C_2\lambda)^2\int_{\mathbb{R}^m}|z|^2e^{(C_2\lambda)^2|z|^2}d\gamma_m(z).\end{aligned}$$

Assume that $2(C_2\lambda)^2<1$. Then, we have

$$\begin{aligned}\int_X e^{\lambda|f(x)|}d\mu_X(x) \leq{}& 1+C_1C_2\lambda(1-2(C_2\lambda)^2)^{-(m+1)/2}\int_{\mathbb{R}^m}|z|d\gamma_m(z) \\ &+C_1(C_2\lambda)^2(1-2(C_2\lambda)^2)^{-(m/2)-1}\int_{\mathbb{R}^m}|z|^2d\gamma_m(z).\end{aligned}$$

By using the Chebyshev inequality, we hence have

$$\begin{aligned}\mu_X(\{x\in X \mid |f(x)|\geq r\}) &\leq e^{-\lambda r}\int_X e^{\lambda|f(x)|}d\mu_X(x) \\ &\leq e^{-\lambda r}\Big\{1+C_1C_2\lambda(1-2(C_2\lambda)^2)^{-(m+1)/2}\int_{\mathbb{R}^m}|z|d\gamma_m(z) \\ &\qquad +C_1(C_2\lambda)^2(1-2(C_2\lambda)^2)^{-(m/2)-1}\int_{\mathbb{R}^m}|z|^2d\gamma_m(z)\Big\}.\end{aligned}$$

Substituting $\lambda := 1/(2C_2\sqrt{m})$ to this inequality, we therefore obtain

$$\begin{aligned}&\mu_X(\{x\in X \mid |f(x)|\geq r\} \\ &\leq e^{-r/(2C_2\sqrt{m})}\Big\{1+\frac{C_1}{2\sqrt{m}}\Big(1-\frac{1}{2m}\Big)^{-(m+1)/2}\int_{\mathbb{R}^m}|z|d\gamma_m(z) \\ &\qquad +\frac{C_1}{4m}\Big(1-\frac{1}{2m}\Big)^{-(m/2)-1}\int_{\mathbb{R}^m}|z|^2d\gamma_m(z)\Big\}.\end{aligned}\tag{3.6}$$

Observe that

$$\int_{\mathbb{R}^m} |z|^2 d\gamma_m(z) = m, \int_{\mathbb{R}^m} |z| d\gamma_m(z) \le \Big(\int_{\mathbb{R}^m} |z|^2 d\gamma_m(z) \Big)^{1/2} = \sqrt{m}, \tag{3.7}$$

and $(1+1/x)^x \le e$ for all $x > 0$. Applying these to (3.6), we obtain the inequality (1.3).

Assume that $p = 2$. We next prove (1.4) in a similar way to the above proof. By virtue of the inequalities (3.2) and (3.4), given $\lambda > 0$, we have

$$\begin{aligned}
\int_X e^{\lambda |f(x)|} d\mu_X(x) &= 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} V_k(f)^k \\
&\le 1 + C_1 C_2 \lambda \sum_{k=1}^{\infty} \frac{(C_2\lambda)^{k-1}}{(k-1)!} \int_{\mathbb{R}^m} |z|^k d\gamma_m(z) \\
&= 1 + C_1 C_2 \lambda \int_{\mathbb{R}^m} |z| e^{C_2 \lambda |z|} d\gamma_m(z).
\end{aligned}$$

Since

$$C_2 \lambda |z| = (\sqrt{2m} C_2 \lambda) \cdot \Big(\frac{|z|}{\sqrt{2m}} \Big) \le m (C_2 \lambda)^2 + \frac{|z|^2}{4m},$$

we calculate that

$$\begin{aligned}
\int_{\mathbb{R}^m} |z| e^{C_2 \lambda |z|} d\gamma_m(z) &\le e^{m(C_2\lambda)^2} \int_{\mathbb{R}^m} |z| e^{\frac{|z|^2}{4m}} d\gamma_m(z) \\
&= e^{m(C_2\lambda)^2} \Big(1 - \frac{1}{2m} \Big)^{-\frac{m+1}{2}} \int_{\mathbb{R}^m} |z| d\gamma_m(z).
\end{aligned}$$

We hence get

$$\begin{aligned}
\mu_X(\{x \in X \mid |f(x)| \ge r\}) &\le e^{-\lambda r} \int_X e^{\lambda |f(x)|} d\mu_X(x) \\
&\le e^{-\lambda r} \Big\{ 1 + C_1 C_2 \lambda e^{m(C_2\lambda)^2} \Big(1 - \frac{1}{2m} \Big)^{-\frac{m+1}{2}} \int_{\mathbb{R}^m} |z| d\gamma_m(z) \Big\}.
\end{aligned}$$

Putting $\lambda := sr/(\sqrt{m} C_2)$ for any $s > 0$, we thus have the estimate

$$\begin{aligned}
\mu_X(\{x \in X \mid |f(x)| \ge r\}) &\le e^{-sr^2/(\sqrt{m}C_2)} \Big\{ 1 + \frac{C_1}{\sqrt{m}} sr e^{s^2 r^2} \Big(1 - \frac{1}{2m} \Big)^{-\frac{m+1}{2}} \int_{\mathbb{R}^m} |z| d\gamma_m(z) \Big\} \\
&\le e^{-sr^2/(\sqrt{m}C_2)} \Big\{ 1 + \frac{C_1}{\sqrt{m}} e^{2s^2 r^2} \Big(1 - \frac{1}{2m} \Big)^{-\frac{m+1}{2}} \int_{\mathbb{R}^m} |z| d\gamma_m(z) \Big\}
\end{aligned}$$

Substituting $s := 1/(4\sqrt{m} C_2)$ into this inequality, we calculate that

$$\mu_X(\{x \in X \mid |f(x)| \ge r\}) \le e^{-r^2/(8m(C_2)^2)} \Big\{ 1 + \frac{C_1}{\sqrt{m}} \Big(1 - \frac{1}{2m} \Big)^{-\frac{m+1}{2}} \int_{\mathbb{R}^m} |z| d\gamma_m(z) \Big\}.$$

As a consequence, by (3.7), we obtain the inequality (1.4). This completes the proof. $\square$

4. Applications and remarks

In this section, we obtain two applications of Corollary 1.2 and compare our results with the results by S. Artstein [1] and Ledoux and Oleszkiewicz [8].

Corollary 4.1. *Let $m \leq n$. Then, for any $r > 0$, we have*

$$\begin{aligned}\mu_{\mathbb{S}^n}(\mathbb{S}^n \setminus (\mathbb{S}^{n-m})_r) \leq \min\{ & A_m e^{-(1/(3\pi))\sqrt{2n/m}r}, \widetilde{A}_m e^{-(1/(3\pi))\sqrt{n/(2m)}r}, \\ & B_m e^{-((n-1)/(4\pi^2 m))r^2}, \widetilde{B}_m e^{-((n-1)/(8\pi^2 m))r^2}\}.\end{aligned} \tag{4.1}$$

Proof. Applying Corollary 1.2 to the projection

$$\mathbb{S}^n \ni (x_1, x_2, \cdots x_{n+1}) \mapsto (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m,$$

we obtain (4.1). This completes the proof. □

The following corollary is a consequence of the theorem of the isoperimetry of waists of the Euclidean sphere by Gromov ([4, Section 1]) and the inequality (4.1).

Corollary 4.2. *Let m and n be two natural numbers such that $m \leq n$ and $f : \mathbb{S}^n \to \mathbb{R}^m$ a continuous map. Then, there exists a point $z_f \in \mathbb{R}^m$ such that*

$$\begin{aligned}\mu_{\mathbb{S}^n}(\mathbb{S}^n \setminus (f^{-1}(z_f))_r) \leq \min\{ & A_m e^{-(1/(3\pi))\sqrt{2n/m}r}, \widetilde{A}_m e^{-(1/(3\pi))\sqrt{n/(2m)}r}, \\ & B_m e^{-((n-1)/(4\pi^2 m))r^2}, \widetilde{B}_m e^{-((n-1)/(8\pi^2 m))r^2}\}\end{aligned}$$

for any $r > 0$.

Let us explain S. Artstein's results for the estimates of the values $\mu_{\mathbb{S}^n}(\mathbb{S}^n \setminus (\mathbb{S}^m)_r)$.

For two variables A and B depending on n, $A \approx B$ means that $\lim_{n\to\infty}(A/B) = 1$. Given $0 < r < \pi/2$ and $0 < \lambda < 1$, we put

$$u(r, \lambda) := (1-\lambda)\log\frac{(1-\lambda)}{\sin^2 r} + \lambda \log\frac{\lambda}{\cos^2 r}.$$

Observe that $u(r, \lambda) \geq 0$ holds for all r, λ.

Theorem 4.3 (cf. [1, Theorem 3.1])**.** *For any $0 < r < \pi/2$ and $0 < \lambda < 1$, the following estimates (1) and (2) both hold as $n \to \infty$.*

(1) *If $\sin^2 r > 1 - \lambda$, then we have*

$$\mu_{\mathbb{S}^n}(\mathbb{S}^n \setminus (\mathbb{S}^{\lambda n})_r) \approx \frac{1}{\sqrt{n\pi}} \frac{\sqrt{\lambda(1-\lambda)}}{\sin^2 r - (1-\lambda)} e^{-\frac{n}{2}u(r,\lambda)}.$$

(2) *If $\sin^2 r < 1 - \lambda$, then we have*

$$\mu_{\mathbb{S}^n}(\mathbb{S}^n \setminus (\mathbb{S}^{\lambda n})_r) \approx 1 - \frac{1}{\sqrt{n\pi}} \frac{\sqrt{\lambda(1-\lambda)}}{\sin^2 r - (1-\lambda)} e^{-\frac{n}{2}u(r,\lambda)}.$$

Theorem 4.4 (cf. [1, Theorem 4.1])**.** *Let $n \geq 6$, $3 \leq m \leq n-3$, and $\lambda := m/n$. Put*

$$l := \frac{\sin^2 r}{1-\lambda} \text{ and } l' := \frac{\cos^2 r}{\lambda}.$$

Then there exist positive constants $c_{n,\lambda}$ and $c'_{n,\lambda}$ both bounded from above by 3 satisfying the following (1) and (2).

(1) *If $\sin^2 r < 1-\lambda$, then*

$$\frac{1}{\sqrt{2\pi}} \frac{e^{-u-c'_{n,\lambda}-\log l'}}{\frac{1}{\sqrt{u+c'_{n,\lambda}+\log l'}} + \sqrt{u+c'_{n,\lambda}+\log l'}} \leq \mu_{\mathbb{S}^n}((\mathbb{S}^m)_r) \leq \frac{1}{\sqrt{2\pi}} \frac{e^{-u-c_{n,\lambda}-\log l}}{\sqrt{u+c_{n,\lambda}+\log l}}.$$

(2) *If $\sin^2 r > 1-\lambda$, then*

$$1 - \frac{1}{\sqrt{2\pi}} \frac{e^{-u-c'_{n,\lambda}-\log l'}}{\frac{1}{\sqrt{u+c'_{n,\lambda}+\log l'}} + \sqrt{u+c'_{n,\lambda}+\log l'}} \leq \mu_{\mathbb{S}^n}((\mathbb{S}^m)_r) \leq 1 - \frac{1}{\sqrt{2\pi}} \frac{e^{-u-c_{n,\lambda}-\log l}}{\sqrt{u+c_{n,\lambda}+\log l}},$$

where $u = \frac{n}{2}\big((1-\lambda)\log\frac{1-\lambda}{\sin^2 r} + \lambda \log \frac{\lambda}{\cos^2 r}\big)$.

Remark 4.5. Fix $0 < \lambda < 1$. By using Theorem 4.3 or Theorem 4.4, we have $\lim_{n\to\infty} \mu_{\mathbb{S}^n}(\mathbb{S}^n \setminus (\mathbb{S}^{\lambda n})_r) = 0$ for all $r > \sin^{-1}\sqrt{1-\lambda}$, which cannot be derived from Corollary 4.1. Theorem 4.3 and 4.4 therefore both contain some information for the values $\mu_{\mathbb{S}^n}((\mathbb{S}^m)_r)$ which Corollary 4.1 does not contain. The author does not know how to derive Corollary 4.1 from Theorems 4.3 and 4.4. However, Corollary 4.1 (and also the inequality (1.6)) is sharp in the following sense. Denote by pr_n the projection from $\mathbb{S}^n(\sqrt{n})$ to the Euclidean space $\mathbb{R}^m$. Since the sequence $\{(\mathrm{pr}_n)_*(\mu_{\mathbb{S}^n(\sqrt{n})})\}_{n=1}^\infty$ of probability measures on $\mathbb{R}^m$ weakly converges to the canonical Gaussian measure γ_m on $\mathbb{R}^m$ (see [6, Lemma 1.2]), by using the inequality (1.6) (or Corollary 4.1), we obtain the estimate

$$\gamma_m(\{x \in \mathbb{R}^m \mid |x| \geq r\}) \leq \min\{B_m e^{-(1/(16m))r^2}, \widetilde{B}_m e^{-(1/(32m))r^2}\}. \tag{4.2}$$

Classically, this inequality was known via an another method, see [9, Section 3.1, (3.5)]. This estimate is sharp in a sense because

$$\lim_{r\to\infty} \frac{\gamma_m(\{x \in \mathbb{R}^m \mid |x| \geq r\})}{e^{-(1/(2m))r^2}} = 1$$

([9, Theorem 3.8]).

Let us compare our result with the inequality (1.7).

Remark 4.6. Combining the inequalities (1.7) with (4.2), we obtain an estimate similar to the inequality (1.3). However, we note that our coefficients of the inequality (1.3) are concrete whereas the coefficients of the inequality (1.7) are not. An advantage of the inequality (1.7) is that we can see from the inequality that the map f concentrates around the expectations if the coefficient C_X is close to zero. This fact cannot be derived from our inequality (1.4). We also remark that their proof cannot be applied to the case where X has the exponential concentration (1.2) (i.e., the case where $p = 1$).

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